International Journal of Applied Information Systems (IJAIS) – ISSN : 2249-0868
Foundation of Computer Science FCS, New York, USA
Volume 12– No.26, December 2019– www.ijais.org

Total Roman Domination in an Interval Graph with Alternate Cliques of Size 3

M. Reddappa
Research Scholar
Dept. of Mathematics,
S.V.University, Tirupati-517502

C. Jaya Subba Reddy
Asst. Professor,
Dept. of Mathematics,
S.V.University, Tirupati-517502

B. Maheswari
Professor(Rtd.)
Dept. of Applied Mathematics,
S.P.M.V.Visvavidyalayam,
Tirupati-517502

ABSTRACT
The theory of Graphs is an important branch of Mathematics that was developed exponentially. The theory of domination in graphs is rapidly growing area of research in graph theory today. It has been studied extensively and finds applications to various branches of Science & Technology.

Interval graphs have drawn the attention of many researchers for over 40 years. They form a special class of graphs with many interesting properties and revealed their practical relevance for modeling problems arising in the real world. The theory of domination in graphs introduced by Ore [12] and Berge [4] has been ever green of graph theory today. An introduction and an extensive overview on domination in graphs and related topics is surveyed and detailed in the two books by Haynes et.al. [1, 2].

In this paper a study of total domination and total Roman domination number of an interval graph with alternate cliques of size 3 is carried out.

Keywords
Total domination number, Total Roman dominating function, Total Roman domination number, Interval family, Interval graph.

1. INTRODUCTION
Domination in graphs has been studied extensively in recent years and it is an important branch of Graph Theory. Allan, R.B. and Laskar, R.C.[3], Cockayne, E.J.and Hedetniemi, S.T [5] and many others have studied various domination parameters of graphs.

Let $G(V,E)$ be a graph. A total dominating set of a graph G with no isolated vertices is a set S of vertices of G such that every vertex in $V(G)$ is adjacent to at least one vertex in S.

The minimum cardinality of a total dominating set is called as total domination number and it is denoted by $\gamma_t(G)$. A total dominating set of G of cardinality $\gamma_t(G)$ is called a $\gamma_t(G)$.-set.

Total domination in graphs was introduced by Cokayne et al. [7]. Total domination is now well studied in graph theory. The literature on the subject of total domination in graphs has been surveyed and detailed in the recent book by Henning et al. [9].

We consider finite graphs without loops and multiple edges.

2. TOTAL ROMAN DOMINATING FUNCTION
The Roman dominating function of a graph G was defined by Cockayne et.al. [6]. The definition of a Roman dominating function was motivated by an article in Scientific American by Ian Stewart [10] entitled "Defend The Roman Empire!" and suggested even earlier by ReVelle [13]. Domination number and Roman domination number in an interval graph with consecutive cliques of size 3 are studied by Jaya Subba Reddy. C, Reddappa. M and Maheswari. B [11].

A Roman dominating function on a graph $G(V,E)$ is a function $f: V \rightarrow \{0,1,2\}$ satisfying the condition that every vertex u for which $f(u) = 0$ is adjacent to at least one vertex v for which $f(v) = 2$. The weight of a Roman dominating function is the value $f(V) = \sum_{v \in V} f(v)$. The minimum weight of a Roman dominating function on a graph G is called as the Roman domination number of G. It is denoted by $\gamma_R(G)$. If $\gamma_R(G) = 2\,\gamma(G)$ then G is called a Roman graph.

Let $f : V \rightarrow \{0, 1, 2\}$ and let (V_0, V_1, V_2) be the ordered partition of V induced by f where $V_i = \{v \in V / f(v) = i\}$ for $i = 0, 1, 2$. Then there exists a 1-1 correspondence between the functions $f : V \rightarrow \{0, 1, 2\}$ and the ordered partition (V_0, V_1, V_2) of V. Thus we write $f = (V_0, V_1, V_2)$.

A function $f = (V_0, V_1, V_2)$ becomes a Roman dominating function if the set V_2 dominates V_0 [4].

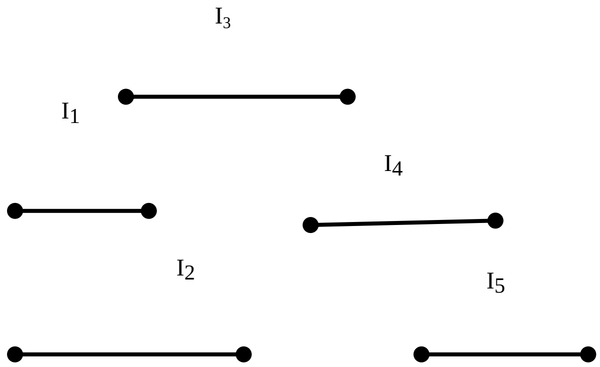
Total Roman domination in graphs are studied by Ahangar et al. [8].A total Roman dominating function of a graph G with no isolated vertices, is a Roman dominating function f on G with the additional property that the sub graph of G induced by the set of all vertices $V_1 \cup V_2$ of positive weight under f has no isolated vertices.

The minimum weight of a total Roman dominating function is called as the total Roman domination number of G and it is denoted by $\gamma_{tR}(G)$. A total Roman dominating function with minimum weight $\gamma_{tR}(G)$ is called $\gamma_{tR}(G)$- function. If $\gamma_{tR}(G) = 2\,\gamma_t(G)$ then G is called a total Roman graph.

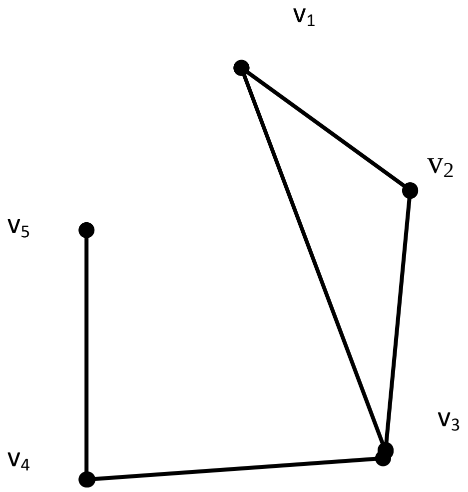
3. INTERVAL GRAPH
Let $I = \{I_1, I_2, I_3, \dots \dots \dots . . I_n\}$ be an interval family, where each I_i is an interval on the real line and $I_i = [a_i, b_i]$ for $i = 1, 2, 3, \dots \dots . . n$. Here a_i is called the left end point and b_i is called the right end point of I_i. Without loss of generality, we assume that all end points of the intervals in I are distinct numbers between 1 and 2n. Two intervals $i = [a_i, b_i]$ and $j = [a_j, b_j]$ are said to intersect each other if either $a_j < b_i$ or $a_i < b_j$. The intervals are labelled in the increasing order of their right end points. Let $G(V,E)$ be a graph. G is called an interval graph if there is a 1-1 correspondence between V and I such that two vertices of G are joined by an edge in E if and

only if their corresponding intervals in I intersect. If i is an interval in I the corresponding vertex in G is denoted by v_i.

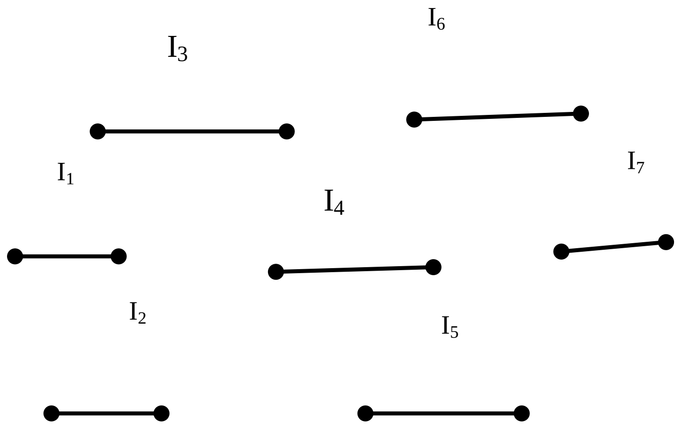
Consider the following interval family.



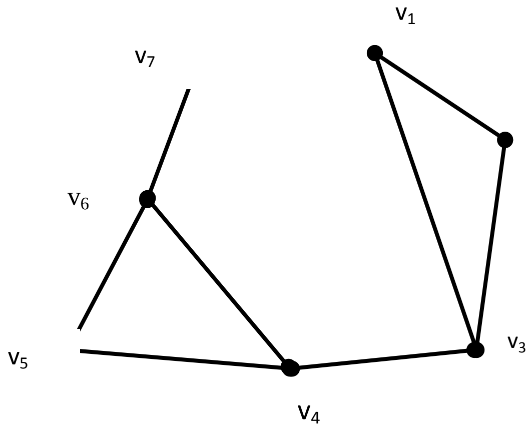
The corresponding interval graph is given by



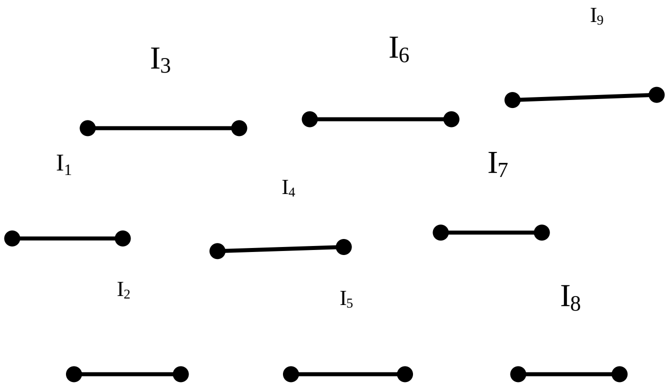
Consider the following interval family.



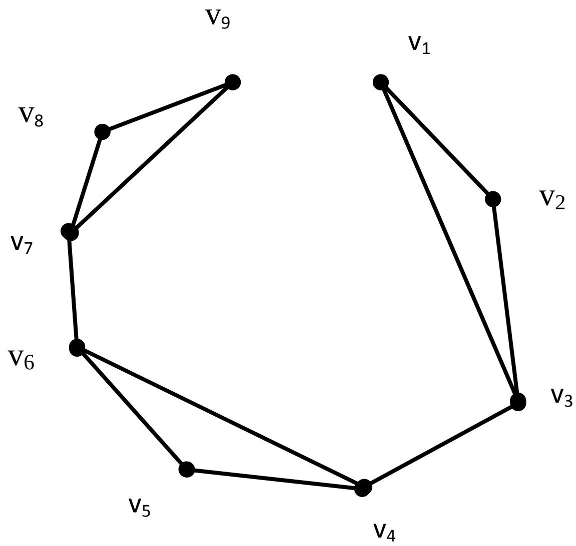
The corresponding interval graph is given by



Consider the following interval family.



The corresponding interval graph is given by



In what follows we consider interval graphs of this type. We observe that when$n = 3k + 3$ then the interval graph has adjacent cliques of size 3, k = 1, 2, 3…… and when $n = 3k + 2$ then the interval graph has adjacent cliques of size 3 and the last clique has two adjacent edges and when $n = 3k + 4$ then the interval graph has adjacent cliques of size 3 and the last clique is adjacent with one edge, k = 1, 2, 3…… .We denote this type of interval graph by $\mathcal{G}$**.** The signed Roman domination is studied in the following for the interval graph $\mathcal{G}$**.**

4. RESULTS

Theorem 4.1: Let $\mathcal{G}$ be the Interval graph of with n vertices and no isolated vertices, where $n \geq 7$. Then the total domination number of $\mathcal{G}$ is

$\gamma_t(\mathcal{G}) = 2k+1$ for $n = 6k+1$

$= 2k+2$ for $n = 6k+2,\ 6k+3, 6k+4, 6k+5, 6k+6$

where $k = 1,2,3 \ldots\ldots$ respectively.

Proof: Let $\mathcal{G}$ be the Interval graph with vertex set $\{v_1, v_2, v_3, v_4 \ldots\ldots\ldots\ldots\ldots v_n\}$ and no isolated vertices, where $n \geq 7$.

Suppose k = 1. Then $n = 7$. We can easily see that $TD = \{v_3, v_4, v_6\}$ is a total dominating set of $\mathcal{G}$. Now for $n = 8, 9$ we see that $TD = \{v_3, v_4, v_6, v_7\}$ and for $n = 10$, $TD = \{v_3, v_4, v_7, v_9\}$ and for $n = 11, 12, TD = \{v_3, v_4, v_9, v_{10}\}$ are total dominating sets of $\mathcal{G}$ respectively. Further we can show that all these sets are minimum total dominating sets. Therefore the total domination numbers of $\mathcal{G}$ are $\gamma_t(\mathcal{G}) = 3$ for $n = 7$ and $\gamma_t(\mathcal{G}) = 4$ for $n = 8, 9, 10, 11, 12$.

If k = 2, then $n = 13, 14, 15, 16, 17, 18$. For$n = 13$, $TD = \{v_3, v_4, v_9, v_{10}, v_{12}\}$ and for $n = 14, 15$, $TD = \{v_3, v_4, v_9, v_{10}, v_{12}, v_{13}\}$ and for $n = 16, TD = \{v_3, v_4, v_9, v_{10}, v_{13}, v_{15}\}$ and for $n = 17, 18, TD = \{v_3, v_4, v_9, v_{10}, v_{15}, v_{16}\}$ are minimum total dominating sets of $\mathcal{G}$. So the total domination numbers are $\gamma_t(\mathcal{G}) = 5$ for $n = 13$ and $\gamma_t(\mathcal{G}) = 6$ for $n = 14, 15, 16, 17, 18$. respectively.

Similarly for k = 3 we have $n = 19, 20, 21, 22, 23, 24$. Then the minimum total dominating sets of $\mathcal{G}$ are

$TD = \{v_3, v_4, v_9, v_{10}, v_{15}, v_{16}, v_{18}\}$ for $n = 19$;

$TD = \{v_3, v_4, v_9, v_{10}, v_{15}, v_{16}, v_{18}, v_{19}\}$ for $n = 20, 21$;

$TD = \{v_3, v_4, v_9, v_{10}, v_{15}, v_{16}, v_{19}, v_{21}\}$ for $n = 22$;

$TD = \{v_3, v_4, v_9, v_{10}, v_{15}, v_{16}, v_{21}, v_{22}\}$ for $n = 23,\ 24$;

respectively.

Hence $\gamma_t(\mathcal{G}) = 7$ for $n = 19$ and $\gamma_t(\mathcal{G}) = 8$ for $n = 20, 21, 22, 23, 24$.

Thus $\gamma_t(\mathcal{G}) = 3$ for $n = 7$

$= 4$ for $n = 8, 9, 10, 11, 12$

$= 5$ for $n = 13$

$= 6$ for $n = 14, 15, 16, 17, 18$

$= 7$ for $n = 19$

$= 8$ for $n = 20, 21, 22, 23, 24$.

Hence we get that the general form of total dominating sets of $\mathcal{G}$ are

$TD = \{v_3, v_4, \ldots\ldots\ldots\ldots\ldots\ldots, v_{n-9},\ v_{n-4}, v_{n-3},\ v_{n-1}\}$ for $n = 7, 13, 19, \ldots\ldots\ldots$

$TD = \{v_3, v_4, \ldots\ldots\ldots\ldots\ldots, v_{n-5},\ v_{n-4},\ v_{n-2},\ v_{n-1}\}$ for $n = 8, 14, 20, \ldots\ldots\ldots$

$TD = \{v_3, v_4, \ldots\ldots\ldots\ldots\ldots, v_{n-6},\ v_{n-5},\ v_{n-3},\ v_{n-2}\}$ for $n = 9, 15, 21, \ldots\ldots\ldots$

$TD = \{v_3, v_4, \ldots\ldots\ldots\ldots\ldots, v_{n-7},\ v_{n-6},\ v_{n-3},\ v_{n-1}\}$ for $n = 10, 16, 22, \ldots\ldots\ldots$

$TD = \{v_3, v_4, \ldots\ldots\ldots\ldots\ldots, v_{n-8},\ v_{n-7},\ v_{n-2},\ v_{n-1}\}$ for $n = 11, 17, 23, \ldots\ldots\ldots$

$TD = \{v_3, v_4, \ldots\ldots\ldots\ldots\ldots, v_{n-9},\ v_{n-8},\ v_{n-3},\ v_{n-2}\}$ for $n = 12, 18, 24, \ldots\ldots\ldots$

and so on.

Thus $\gamma_t(\mathcal{G}) = 2k+1$ for $n = 6k+1$

$= 2k+2$ for $n = 6k+2,\ 6k+3, 6k+4, 6k+5, 6k+6$

where $k = 1,2,3 \ldots\ldots$ respectively.

Theorem 4.2: Let $\mathcal{G}$ be the interval graph with n vertices and no isolated vertices, where $2 < n < 7$. Then $\gamma_t(\mathcal{G}) = 2$.

Proof: Let $\mathcal{G}$ be the interval graph with n vertices and no isolated vertices, where $2 < n < 7$.

Let $\{v_1, v_2, v_3, v_4, v_5, v_6\}$ be the vertices of $\mathcal{G}$.

Then it is clear that $\{v_2, v_3\}$ is the total dominating set when$n$ = 3 and $\{v_3, v_4\}$ is the total dominating set when n = 4, 5, 6.

That is $\gamma_t(\mathcal{G}) = 2$.

Theorem 4.3: Let $\mathcal{G}$ be the interval graph with n vertices and no isolated vertices, where $n \geq 7$. Then the total Roman domination number of $\mathcal{G}$ is

$\gamma_{tR}(\mathcal{G}) = 4k+2$ for $n = 6k+1$

$= 4k+4$ for $n = 6k+2,\ 6k+3,\ 6k+4, 6k+5, 6k+6$

where $k = 1,2,3 \ldots\ldots$ respectively.

Proof: Let $\mathcal{G}$ be the interval graph with n vertices and no isolated vertices, where $n \geq 7$.

Let the vertex set $\mathcal{G}$ of be $\{v_1, v_2, v_3, v_4 \ldots\ldots\ldots\ldots\ldots v_n\}$.

Case 1: Suppose $n = 6k+1$, where $k = 1,2,3 \ldots\ldots$.

Let $f : V \to \{0, 1, 2\}$ and let (V_0, V_1, V_2) be the ordered partition of V induced by f where $V_i = \{v \in V / f(v) = i\}$ for $i = 0, 1, 2$. Then there exist a 1-1 correspondence between the functions $f : V \to \{0, 1, 2\}$ and the ordered pairs (V_0, V_1, V_2) of V.Thus we write$f = (V_0, V_1, V_2)$.

Let$V_1 = \{\emptyset\}$;
$V_2 = \{v_3, v_4, \ldots\ldots\ldots\ldots\ldots, v_{n-9}, v_{n-4}, v_{n-3}, v_{n-1}\}$;

V_0=V-$\{V_2\}$= $\{v_1, v_2, v_5, \ldots\ldots\ldots\ldots\ldots v_{n-5}, v_{n-2}, v_n\}$.

By Theorem 4.1, we see that V_2 is a minimum total dominating set of $\mathcal{G}$. Further the setV_2 dominates V_0. In addition the induced sub graph on $V_1 \cup V_2$ is a sub graph of $\mathcal{G}$ with no isolated vertices.

Therefore $f = (V_0, V_1, V_2)$ is a total Roman dominating function of $\mathcal{G}$.

Now $|V_2| = 2k+1, |V_1| = 0, |V_0| = n - (2k+1)$.

Therefore $\sum_{v \in V} f(v) = \sum_{v \in V_0} f(v) + \sum_{v \in V_1} f(v) + \sum_{v \in V_2} f(v)$

$2(2k+1)$ = $4k$+2

Let $g = (V_0',\ V_1',\ V_2')$ be a total Roman dominating function of $\mathcal{G}$, where V_2' dominates V_0'. Then

$$g(V) = \sum_{v\in V'} g(v) = \sum_{v\in V_0'} g(v) + \sum_{v\in V_1'} g(v) + \sum_{v\in V_2'} g(v)$$

$$= |V_1'| + 2|V_2'|$$

Since V_2 is a minimum dominating set of $\mathcal{G}$, we have $|V_2| \le |V_2'|$. Further $|V_1'| > |V_1|$, since $|V_1| = 0$. This implies that $g(V) = |V_1'| + 2|V_2'| > |V_1| + 2|V_2| = f(V)$. Thus $f(V)$ is the minimum weight of $\mathcal{G}$, where $f(V_0, V_1, V_2)$ is a total Roman dominating function.

Therefore $\gamma_{tR}(G) = 4k + 2$.

Case 2: Suppose $n = 6k + 2$, where $k = 1,2,3 \dots\dots$.

Now we proceed as in Case 1.

Let $V_1 = \{\emptyset\}$,

$V_2 = \{v_3, v_4, \dots\dots\dots\dots, v_{n-5}, v_{n-4}, v_{n-2}, v_{n-1}\}$,

$V_0 = V - \{V_2\} = \{v_1, v_2, v_5, \dots\dots\dots\dots v_{n-6}, v_{n-3}, v_n\}$.

Clearly V_2 is a minimum total dominating set of $\mathcal{G}$. Here we observe that the set V_2 dominates V_0.

Therefore $f = (V_0, V_1, V_2)$ becomes a total Roman dominating function of $\mathcal{G}$.

Now $|V_2| = 2k + 2, |V_1| = 0, |V_0| = n - (2k + 2)$.

Therefore $\sum_{v\in V} f(v) = \sum_{v\in V_0} f(v) + \sum_{v\in V_1} f(v) + \sum_{v\in V_2} f(v)$
$= 2(2k + 2) = 4k+4$.

Let $g = (V_0', V_1', V_2')$ be a total Roman dominating function of $\mathcal{G}$. Then we can show as in Case 1, that $f(V)$ is a minimum weight of $\mathcal{G}$ for the total Roman dominating function $f(V_0, V_1, V_2)$.

Thus $\gamma_{tR}(\mathcal{G}) = 4k + 4$.

Case 3: Suppose $n = 6k + 3$, where $k = 1,2,3 \dots\dots$.

Now we proceed as in Case 1.

Let $V_1 = \{\emptyset\}$, $V_2 = \{v_3, v_4, \dots\dots\dots\dots, v_{n-6}, v_{n-5}, v_{n-3}, v_{n-2}\}$, $V_0 = V - \{V_2\} = \{v_1, v_2, v_5, \dots\dots\dots\dots, v_{n-4}, v_{n-1}, v_n\}$.

Obviously V_2 is a minimum total dominating set of $\mathcal{G}$. Here we observe that the set V_2 dominates V_0.

Therefore $f = (V_0, V_1, V_2)$ becomes a total Roman dominating function of $\mathcal{G}$.

Now $|V_2| = 2k + 2, |V_1| = 0, |V_0| = n - (2k + 2)$.

Therefore $\sum_{v\in V} f(v) = \sum_{v\in V_0} f(v) + \sum_{v\in V_1} f(v) + \sum_{v\in V_2} f(v)$
$=2(2k + 2) = 4k+4$.

Let $g = (V_0', V_1', V_2')$ be a total Roman dominating function of $\mathcal{G}$. Then it follows as in Case 1, that $f(V)$ is a minimum weight of $\mathcal{G}$ for the total Roman dominating function $f(V_0, V_1, V_2)$.

Thus $\gamma_{tR}(\mathcal{G}) = 4k + 4$.

Case 4: Suppose $n = 6k + 4$, where $k = 1,2,3 \dots\dots$.

Now we proceed as in Case 1.

Let $V_1 = \{\emptyset\}$,
$V_2 = \{v_3, v_4, \dots\dots\dots\dots, v_{n-7}, v_{n-6}, v_{n-3}, v_{n-1}\}$,

$V_0 = V - \{V_2\} = \{v_1, v_2, v_5, \dots\dots\dots\dots, v_{n-4}, v_{n-2}, v_n\}$.

Again V_2 is a minimum total dominating set of $\mathcal{G}$ and the set V_2 dominates V_0.

Therefore $f = (V_0, V_1, V_2)$ becomes a total Roman dominating function of $\mathcal{G}$.

Now $|V_2| = 2k + 2, |V_1| = 0, |V_0| = n - (2k + 2)$.

Therefore $\sum_{v\in V} f(v) = \sum_{v\in V_0} f(v) + \sum_{v\in V_1} f(v) + \sum_{v\in V_2} f(v)$
$=2(2k + 2) = 4k+4$.

Let $g = (V_0', V_1', V_2')$ be a total Roman dominating function of $\mathcal{G}$. In similar lines to Case 1, we can show that $f(V)$ is a minimum weight of $\mathcal{G}$ for the total Roman dominating function $f(V_0, V_1, V_2)$.

Thus $\gamma_{tR}(\mathcal{G}) = 4k + 4$.

Case 5: Suppose $n = 6k + 5$, where $k = 1,2,3 \dots\dots$.

Now we proceed as in Case 1.

Let $V_1 = \{\emptyset\}$,
$V_2 = \{v_3, v_4, \dots\dots\dots\dots, v_{n-8}, v_{n-7}, v_{n-2}, v_{n-1}\}$,

$V_0 = V - \{V_2\} = \{v_1, v_2, v_5, \dots\dots\dots\dots, v_{n-4}, v_{n-3}, v_n\}$.

Here V_2 is a minimum total dominating set of $\mathcal{G}$ and the set V_2 dominates V_0.

Therefore $f = (V_0, V_1, V_2)$ becomes a total Roman dominating function of $\mathcal{G}$.

Now $|V_2| = 2k + 2, |V_1| = 0, |V_0| = n - (2k + 2)$.

Therefore $\sum_{v\in V} f(v) = \sum_{v\in V_0} f(v) + \sum_{v\in V_1} f(v) + \sum_{v\in V_2} f(v)$
$=2(2k + 2) = 4k+4$.

Let $g = (V_0', V_1', V_2')$ be a total Roman dominating function of $\mathcal{G}$. Then we can show as in Case 1, that $f(V)$ is a minimum weight of $\mathcal{G}$ for the total Roman dominating function $f(V_0, V_1, V_2)$.

Thus $\gamma_{tR}(\mathcal{G}) = 4k + 4$.

Case 6: Suppose $n = 6k + 6$ **,** where $k = 1,2,3 \dots\dots$.

Now we proceed as in Case 1.

Let $V_1 = \{\emptyset\}$,
$V_2 = \{v_3, v_4, \dots\dots\dots\dots, v_{n-9}, v_{n-8}, v_{n-3}, v_{n-2}\}$,

$V_0 = V - \{V_2\} = \{v_1, v_2, v_5, \dots\dots\dots, v_{n-4}, v_{n-1}, v_n\}$.

Clearly V_2 is a minimum total dominating set of $\mathcal{G}$. Further the set V_2 dominates V_0.

Therefore $f = (V_0, V_1, V_2)$ becomes a total Roman dominating function of $\mathcal{G}$.

Now $|V_2| = 2k + 2, |V_1| = 0, |V_0| = n - (2k + 2)$.

Therefore $\sum_{v\in V} f(v) = \sum_{v\in V_0} f(v) + \sum_{v\in V_1} f(v) + \sum_{v\in V_2} f(v)$
$=2(2k + 2) = 4k+4$.

Let $g = (V_0', V_1', V_2')$ be a total Roman dominating function of $\mathcal{G}$. Then we can show as in Case 1, that $f(V)$ is a minimum weight of $\mathcal{G}$ for the total Roman dominating function $f(V_0, V_1, V_2)$.

Thus $\gamma_{tR}(\mathcal{G}) = 4k + 4$.

Theorem 4.4: Let $\mathcal{G}$ be the interval graph of with *n* vertices and no isolated vertices, where $2 < n < 7$. Then the total Roman domination number is

$\gamma_{tR}(\mathcal{G}) = 3$ for $n = 3, 4$

= 4 for $n = 5, 6$.

Proof: Let $\mathcal{G}$ be the interval graph of with *n* vertices and no isolated vertices, where $2 < n < 7$.Let $\{v_1, v_2, v_3, v_4, v_5, v_6\}$ be the vertices of $\mathcal{G}$.

Case 1: Suppose $n = 3$. Let v_1, v_2, v_3 be the vertices of $\mathcal{G}$.

Let $V_1 = \{v_3\}$; $V_2 = \{v_2\}$; $V_0 = V - \{V_1 \cup V_2\} = \{v_1\}$.

We observe that $V_1 \cup V_2$ is a minimum total dominating set of $\mathcal{G}$ and the set V_2 dominates V_0. In addition the induced sub graph on $V_1 \cup V_2$ is a sub graph of $\mathcal{G}$ with no isolated vertices.

Therefore $f = (V_0, V_1, V_2)$ is a total Roman dominating function of $\mathcal{G}$.

Therefore

$$\sum_{v \in V} f(v) = \sum_{v \in V_0} f(v) + \sum_{v \in V_1} f(v) + \sum_{v \in V_2} f(v).$$

= 0 + 1+2 = 3

Thus $\gamma_{tR}(\mathcal{G}) = 3$.

Case 2: Suppose $n = 4$. Let v_1, v_2, v_3, v_4 be the vertices of $\mathcal{G}$.

Let $V_1 = \{v_4\}$; $V_2 = \{v_3\}$; $V_0 = V - \{V_1 \cup V_2\} = \{v_1, v_2\}$.

Clearly $V_1 \cup V_2$ is a minimum total dominating set of $\mathcal{G}$ and the set V_2 dominates V_0. Now we proceed as in Case 1,and hence we have $\gamma_{tR}(\mathcal{G}) = 3$.

Case 3: Suppose $n = 5$. Let v_1, v_2, v_3, v_4, v_5 be the vertices of $\mathcal{G}$.

Let $V_1 = \{\emptyset\}$; $V_2 = \{v_3, v_4\}$; $V_0 = V - \{V_2\} = \{v_1, v_2, v_5\}$.

We observe that V_2 is a minimum total dominating set of $\mathcal{G}$ and the set V_2 dominates V_0.

Therefore $f = (V_0, V_1, V_2)$ is a total Roman dominating function of $\mathcal{G}$.

Therefore

$$\sum_{v \in V} f(v) = \sum_{v \in V_0} f(v) + \sum_{v \in V_1} f(v) + \sum_{v \in V_2} f(v).$$

= 0 +2 x 2 = 4

Thus $\gamma_{tR}(\mathcal{G}) = 4$.

Case 4: Suppose $n = 6$. Let $v_1, v_2, v_3, v_4, v_5, v_6$ be the vertices of $\mathcal{G}$.

Let $V_1 = \{\emptyset\}$; $V_2 = \{v_3, v_4\}$; $V_0 = V - \{V_2\} = \{v_1, v_2, v_5, v_6\}$.

Clearly V_2 is a minimum total dominating set of $\mathcal{G}$ and the set V_2 dominates V_0. Now we proceed as in Case 4, and hence we have $\gamma_{tR}(\mathcal{G}) = 4$.

Theorem 4.5: Let $\mathcal{G}$ be the interval graph with *n* vertices and no isolated vertices, where $2 < n < 5$. Then$\gamma_{tR}(\mathcal{G}) = \gamma_t(\mathcal{G}) + 1$.

Proof : Let $\mathcal{G}$ be the interval graph with *n* vertices and no isolated vertices, where $2 < n < 5$.

Then it is clear that when $n = 3, 4$, from Theorem 4.4 and Theorem 4.2 we get

$$\gamma_{tR}(\mathcal{G}) = 3 = 2 + 1 = \gamma_t(\mathcal{G}) + 1.$$

Thus $\gamma_{tR}(\mathcal{G}) = \gamma_t(\mathcal{G}) + 1$.

Theorem 4.6: Let $\mathcal{G}$ be the interval graph with *n* = 6 vertices and no isolated vertices. If $\gamma(\mathcal{G}) = \gamma_t(\mathcal{G})$, then $\mathcal{G}$ is a total Roman graph.

Proof: Let $\mathcal{G}$ be the interval graph of with *n* = 6 vertices and no isolated vertices.

Suppose *n* = 6.

Then we have $\gamma(\mathcal{G}) = 2$ and $\gamma_t(\mathcal{G}) = 2$.

Thus $\gamma(\mathcal{G}) = \gamma_t(\mathcal{G})$.

Therefore $\mathcal{G}$ is a total Roman graph.

Theorem 4.7: Let $\mathcal{G}$ be the interval graph with *n* vertices and no isolated vertices, where $n \geq 7$. Then $\mathcal{G}$ is a total Roman graph, for $n = 6k + 1, 6k + 2, 6k + 3, 6k + 4, 6k + 5, 6k + 6$

where $k = 1, 2, 3, \ldots$ ....respectively.

Proof: Let $\mathcal{G}$ be the interval graph with n vertices and no isolated vertices, where $n \geq 7$. Where $n = 6k + 1, 6k + 2, 6k + 3, 6k + 4, 6k + 5, 6k + 6$ and

$k = 1, 2, 3, \ldots$ ....respectively.

Case 1: Suppose $= 6k + 1$, and

$k = 1, 2, 3, \ldots$ ....respectively.

Then by Theorem 4.3, we have the total Roman domination number is

$\gamma_{tR}(\mathcal{G}) = 4k + 2$ for $n = 6k + 2, 6k + 3$.

$= 2(2k + 1) = 2\gamma_t(\mathcal{G})$

Thus $\mathcal{G}$ is a total Roman graph.

Case 2: Suppose $n = 6k + 2, 6k + 3, 6k + 4, 6k + 5, 6k + 6$, and $k = 1, 2, 3, \ldots$ ....respectively.

Then by Theorem 4.3, we have the total Roman domination number is

$\gamma_{tR}(\mathcal{G}) = 4k + 4$ for$n = 6k + 2, 6k + 3, 6k + 4, 6k + 5, 6k + 6$,

$= 2(2k + 2) = 2\gamma_t(\mathcal{G})$

Therefore $\mathcal{G}$ is a total Roman graph.

Theorem 4.8: Let $\mathcal{G}$ be the interval graph with *n* vertices and no isolated vertices, where $n \geq 7$. Then $\mathcal{G}$ is a total Roman graph if and only if there exist a γ_{tR-}function $f = (V_0, V_1, V_2)$ with $|V_1| = 0$.

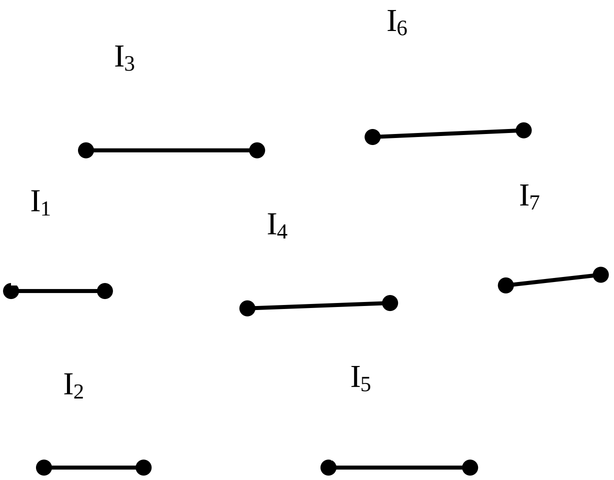
Proof: Let $\mathcal{G}$ be the interval graph with n vertices and no isolated vertices, where $n \geq 7$. Suppose $\mathcal{G}$ is a total Roman graph. Let $f = (V_0, V_1, V_2)$ be a γ_{tR-}function of $\mathcal{G}$. Then we know that V_2 dominates V_0 and $V_1 \cup V_2$ dominates V. In addition the induced sub graph $V_1 \cup V_2$ is a sub graph of $\mathcal{G}$ with no isolated vertices. Hence $\gamma_t(\mathcal{G}) \leq |V_1 \cup V_2| = |V_1| + |V_2| \leq |V_1| + 2|V_2| = \gamma_{tR}(\mathcal{G})$. But $\mathcal{G}$ is a total Roman graph. So $\gamma_{tR}(\mathcal{G}) = 2\,\gamma_t(\mathcal{G})$. Then it follows that $|V_1| = 0$, which establishes Theorem 4.3.

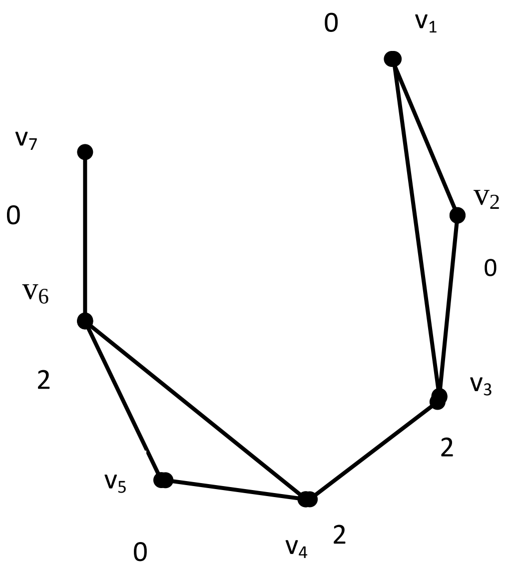
Conversely, suppose there is aγ_{tR-}function $f = (V_0, V_1, V_2)$ of $\mathcal{G}$ such that $|V_1| = 0$. By the definition of γ_{tR-}function, we have $V_1 \cup V_2$ dominates V and since $|V_1| = 0$, it follows that V_2 dominates V. In addition the induced sub graph $V_1 \cup V_2$ is a sub graph of $\mathcal{G}$ with no isolated vertices. As V_2 is a minimum total dominating set, we have $\gamma_t(\mathcal{G})= |V_2|$. By the definition of γ_{tR-}function we have $\gamma_{tR}(\mathcal{G}) = |V_1| + 2|V_2| = 0 + 2|V_2| = 2\,\gamma_t(\mathcal{G})$.

Hence $\mathcal{G}$ is a total Roman graph, which also establishes Theorem 4.3

5.ILLUSTRATIONS

Illustration 1: $n = 7$





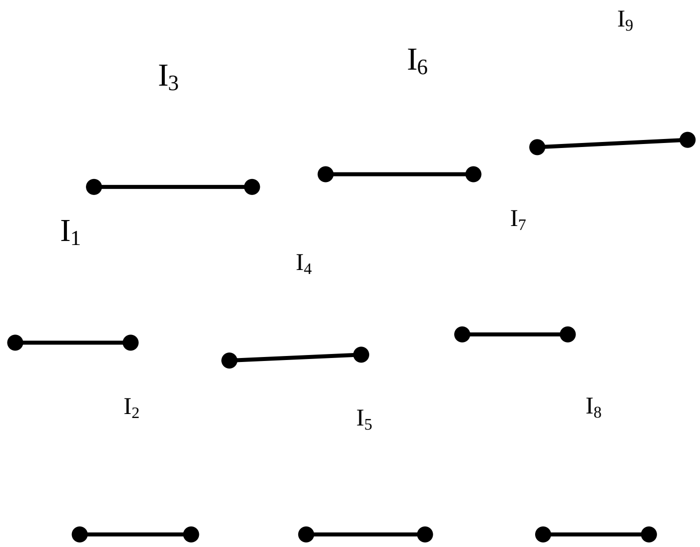
$TD = \{v_3, v_4, v_6\}$ and $\gamma_t(\mathcal{G}) = 3$.

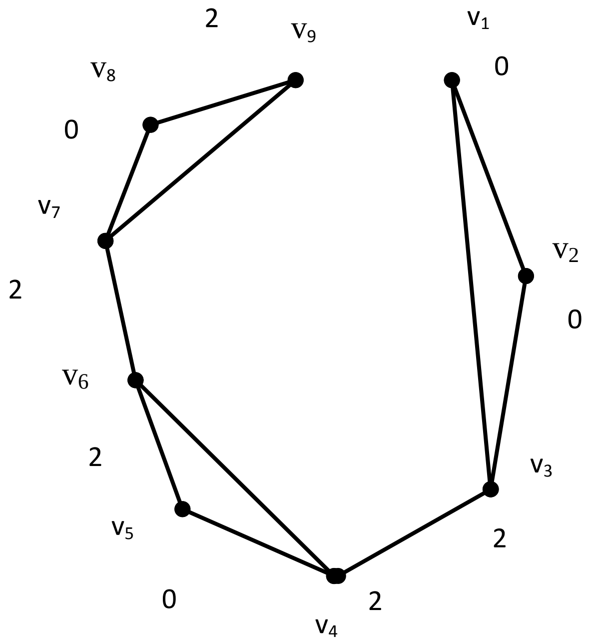
$V_1 = \{\emptyset\}$; $V_2 = \{v_3\,, v_4, v_6\}$; V_0= V-$\{V_2\}$= $\{v_1, v_2, v_5, v_7\}$

$$\sum_{v \in V} f(v) = |V_1| + 2|V_2| = 0 + 2 \times 3 = 6 = f(V)$$

Therefore $\gamma_{tR}(\mathcal{G}) = 6$.

Illustration 2: $n = 9$





$TD = \{v_3, v_4, v_7, v_9\}$ and $\gamma_t(\mathcal{G}) = 4$.

$V_1 = \{\emptyset\}$;$V_2 = \{v_3, v_4, v_7, v_9\}$;$V_0$=V-$\{V_2\}$= $\{v_1, v_2,, v_5, v_6, v_8, v_{10}\}$

$$\sum_{v \in V} f(v) = |V_1| + 2|V_2| = 0 + 2 \times 4 = 8 = f(V)$$

Therefore $\gamma_{tR}(\mathcal{G}) = 8$.

5. REFERENCES

[1] Haynes, T.W., Hedetniemi, S.T., and Slater, P.J. 1998. Domination in graphs: Advanced Topics, Marcel Dekkar, Inc., New York.

[2] Haynes, T.W., Hedetniemi, S.T., and Slater, P.J.1998. Fundamentals of domination in graphs, Marcel Dekkar, Inc., New York.

[3] Allan, R.B. and Laskar, R.C. 1978. On domination, Independent domination numbers of a graph Discrete Math., 23, 73-76.

[4] Berge, C. 1980. Graphs and Hyperactive graphs, North Holland, Amsterdam in graphs, Networks, 10,211 – 215.

[5] Cockayne, E.J. and Hedetniemi, S.T. 1977. Towards a theory of domination in graphs. Networks 7,247 -261.

[6] Cockayne, E.J. Dreyer, P.A., Hedetniemi, S.M., and Hedetniemi, S.T. 2004. Roman domination in graphs, Discrete math., 278, 11 -22.

[7] Cockayne, E. J., Dawes, R.M., Hedetniemi, S.T. 1980. Total domination in graphs, Networks, 10, 211-219.

[8] Hossein Abdollahzadeh Ahangar, Michal A. Henning, Vladimir Samodivkin, Ismael G Yero 2016. Total Roman domination in graphs, Appl. Anal. Discrete math. 10,501-517.

[9] Henning, M.A., Yeo A. 2013. Total domination in graphs.(Springer Mono graphs in Mathematics) ISBN:978-1-4614-6524-9 (Print) 978-1-4614-6525-6(Online).

[10] Ian Stewart. 1999. Defend the Roman Empire!., Scientific American, 281(6), 136 -139.

[11] Jaya Subba Reddy. C., Reddappa, M., and Maheswari. B. 2019. Roman domination in a certain type of interval graph, International Journal of Research and analyticalReviews, 6(1), 665–672.

[12] Ore, O. 1962. Theory of Graphs, Amer, Math.Soc. Collaq.Publ.38, Providence.

[13] ReValle, C.S., and Rosing K,E. 2000. Defendens imperium romanum: a classical problem in military Strategy, Amer. Math. Monthly, 107 (7), 585 -594.